This is a short handout discussing quotient sets for Math 55.

1 Quotient Sets

Quotient sets allow us to "identify elements" in a set A, in the sense that we create a new set A' that "lumps all identified elements into a single element." The construction is ubiquitous in all areas of math, and usually other structures (algebraic, topological, etc.) are placed on the set. The resulting "quotient set" will inherit many of the structures of the parent set, but will allow us to treat identified elements as equal.

Let's now make the vague discussion above more precise. First, recall the definition of an equivalence relation:

Definition 1.1. An *equivalence* relation $\sim$ on a set A is a relation on A that is reflexive, symmetric, and transitive.

The conditions mean that for all $a, b, c \in A$, $a \sim a$, $a \sim b$ implies $b \sim a$, and $a \sim b$ and $b \sim c$ imply $a \sim c$.

Definition 1.2. For $a \in A$, we define the *equivalence class* of A to be the subset $[a] := \{b \in A : b \sim a\}$.

Notice that $[a]$ is never empty, because $a \in [a]$ by reflexivity. Hence $\bigcup_{a \in A}[a] = A$. But this is not just any union of subsets, it is a *partition*:

Proposition 1.1. The equivalence classes of $[a]$ partition A. In other words, they union to A, and for any two equivalence classes $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Proof. The first statement was proved above. For the second statement, suppose $[a] \cap [b] \neq \emptyset$, so there is a common element b in both. Then $a \sim c$ and $c \sim b$ (using symmetry), so $a \sim b$. Therefore

$$d \in [a] \Leftrightarrow d \sim a \Leftrightarrow d \sim b \Leftrightarrow d \in [b]$$

by various applications of symmetry and transitivity of the equivalence relation, so $[a] = [b]$. $\square$

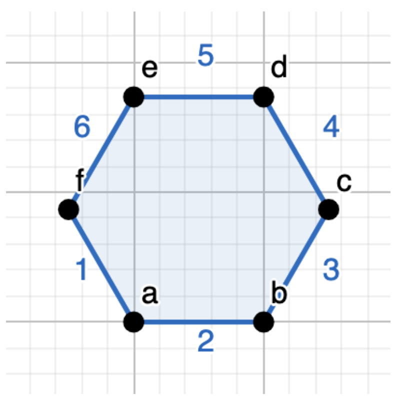
So every element in A is in one, and exactly one, equivalence class. Moreover, for any equivalence class C and $a \in C$, we have $[a] = C$. In that case, we call a a *representative* of C. Of course, a given class C can have many different representatives. If $\{a_i\}_{i \in I}$ is a subset of A such that every equivalence class is equal to $[a_i]$ for *exactly one* a_i, then we call the $\{a_i\}$ a *system of representatives* for the equivalence classes.

Example 1.1. Let $A = \mathbf{Z}$, let $n \geq 2$ be an integer, and $\sim$ be the equivalence relation $a \sim b$ if and only if $n|(a-b)$. In other words, $a \sim b$ if and only if $a \equiv b \bmod n$. The equivalence classes of this equivalence relation are exactly the $[0], [1], \ldots, [n-1]$. Hence $\{0, 1, \ldots, n-1\}$ form a system of representatives. Any element in $[0]$, like $10000n$, is a representative for $[0]$, so a system of representatives could also be $\{n^1 + 0, n^2 + 1, n^3 + 2, \ldots, n^n + (n-1)\}$.

Example 1.2. Let A be the set of all circles in the plane, and $\sim$ the equivalence relation $C_1 \sim C_2$ if and only if C_1 and C_2 are congruent. Then any equivalence class is represented by a unique circle with center at the origin, so these form a system of representatives.

Example 1.3. Let $A = \mathbf{Z} \times (\mathbf{Z} - \{0\})$, and let $\sim$ be the equivalence relation $(a, b) \sim (c, d)$ if and only if $ad = bc$. Then the set $\{(a, b) : b > 0, \gcd(a, b) = 1\}$ is a system of representatives for the equivalence classes of $\sim$.

Example 1.4. Let A be the set of all *colorings of* vertices of a hexagon, where each vertex is given a color in the set {red, blue, yellow}. Let $\sim$ be the equivalence relation on A such that two colorings are equivalent if and only if they can be superimposed upon each other by a rotation or reflection of the hexagon. For example, in the below hexagon, the colorings $\{R, B, Y, R, R, Y\}$ and $\{B, Y, R, R, Y, R\}$ (starting from vertex a and going counterclockwise) are equivalent, because they can be superimposed on each other by a 60-degree rotation. The question of finding a system of representatives for the equivalence classes of $\sim$ is more or less answered by *Burnside's Lemma* from group theory.



We can now define what a quotient set is.

Definition 1.3. Let A be a set and $\sim$ an equivalence relation on A. Then the quotient set $A/\sim$ is the set of all equivalence classes of $\sim$. We read this as "the quotient set of A by $\sim$", or "Amod $\sim$" (Example 1.5 will give a good reason for this name).

Intuitively (and literally), this means that all elements of a single equivalence class are treated as a single element in the quotient. Although this idea may seem a bit strange at first, we'll see in the next few examples that you already know many quotient sets.

The quotient comes with a "natural" map from its "parent" set: $\pi : A \to A/\sim$ is given by $a \mapsto [a]$. We call this the *canonical projection* map, and it is surjective by construction. The fibers of this surjection are precisely the equivalence classes under $\sim$.

Oftentimes, if the equivalence relation on A is somewhat reasonable (e.g. defined by some algebraic relation), then $A/\sim$ should have some "natural" bijection to a more familiar set. If you're familiar with basic algebraic structures, such as groups, rings, vector spaces, etc., then these "natural bijections" are just group/ring/vector space/etc. isomorphisms. If not, then the following examples are good warm-ups for things you'll see later on in future classes.

Example 1.5. Recall the equivalence relation on $\mathbf{Z}$ from Example 1.1. Then $\mathbf{Z}/\sim$ is in natural bijection with the set $\mathbf{Z}/n\mathbf{Z}$ that we studied earlier in class, since the bijection $\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/\sim$ is given by $a \mapsto [a]$. The canonical projection $\mathbf{Z} \to \mathbf{Z}/\sim$ is given by $a \mapsto [a]$ (note that these are different maps because their domains are different!). The aforementioned bijection and projection preserve the "addition and multiplication operations" on the relevant sets (we define addition on $\mathbf{Z}/\sim$ by $[a]+[b] = [a+b]$, and multiplication similarly), so these are examples of group isomorphisms/homomorphisms.

Example 1.6. Recall the equivalence relation on A, the set of all circles, from Example 1.2. Then $A/\sim$ is in natural bijection with the positive real interval $(0, \infty)$. Indeed, the equivalence class of a circle is completely defined by its radius $r > 0$: all circles of the same radius are congruent (so in the same equivalence class), while circles of different radii are not (so in different equivalence classes). Therefore the bijection from $(0, \infty) \to A/\sim$ is given by $r \mapsto [B_r]$, where B_r is the circle of radius r centered at the origin. This is a very basic example of an abstract idea in algebraic geometry called a *moduli space*, which is a type of "classifying space" (like how the positive real line "classifies" circles, up to congruence).

Example 1.7. Recall the equivalence relation on $A = \mathbf{Z} \times (\mathbf{Z} - \{0\})$ from Example 1.3. Then $A/\sim$ is in natural bijection with $\mathbf{Q}$. Indeed, every equivalence class $[(a, b)]$ determines a unique rational number a/b, because by definition of the equivalence relation ($(a, b) \sim (c, d)$ iff $ad = bc$), the ordered pairs (c, d) in $[(a, b)]$ are exactly those such that $a/b = c/d$. So the bijection $(A/\sim) \to \mathbf{Q}$ is given by $[(a, b)] \mapsto a/b$, which is easily verified to be a bijection (good exercise).

2 Well-definedness of functions

One common pitfall when dealing with quotient sets is the possibility for a function to be *ill-defined*, or *not well-defined*. To get intuition for this concept, it's probably easier to see a few examples first. The common theme to both examples is looking at what can go wrong when there are possibly *multiple ways of expressing* the input to a "function."

Example 2.1. Consider the "function" $\mathbf{Q} \to \mathbf{Z}$, putatively defined by $a/b \to a + b$ (where the numerator a and denominator b are integers). The issue is that this function *does not actually make sense*. For instance, suppose we both want to plug in "one-half" into this "function". I write "one-half" as $1/2$, plug it in, and get $1 + 2 = 3$. You write "one-half" as $(-2)/(-4)$, plug it in, and get $-2 + (-4) = -6$. This is a huge problem: we plugged in the *exact same element of the set* $\mathbf{Q}$, but got two different outputs, contrary to the requirements for a function (which must have *exactly one output for each input*). So our function is not well-defined, because *different representations of the same input don't give the same output.*

Example 2.2. Let $\sim$ be the equivalence relation on $\mathbf{Z}$ given by $a \sim b$ iff $5|(a-b)$ (the "mod 5 equivalence relation"), and let $\sim'$ be the equivalence relation on $\mathbf{Z}$ given by $a \sim b$ iff $3|(a-b)$. Consider the "function" $(\mathbf{Z}/\sim) \to (\mathbf{Z}/\sim')$, putatively defined by $[a] \mapsto [a]'$ (with the prime denoting equivalence classes in $(\mathbf{Z}/\sim')$). That is, we try to map "an equivalence class mod 5" to to "an equivalence class mod 3". Again, this function is not well-defined. For instance, suppose we both want to plug in the element $[1]$ of $\mathbf{Z}/\sim$ into this "function". I write this element as $[1]$, plug it in, and get $[1]'$ by the given rule. You write this element as $[6]$ (as $[6] = [1]$ in $(\mathbf{Z}/\sim)$!), plug it in, and get $[6]' = [0]' \in (\mathbf{Z}/\sim')$ by the given rule. But $[1]' \neq [0]'$ in $(\mathbf{Z}/\sim')$, so again we have a problem. In particular, the problem is that we have multiple ways to represent the same element in $\mathbf{Z}/\sim$, and those multiple representations might give different results upon applying the same "numerical rule" to those representations.

So in order for a "rule" to actually be a *function*, part of the definition is that it must produce a unique output for every input. In particular, that output *cannot change if my representation of the input is different, but the actual input (as an element of the domain) stays the same*. This is what it means for a function to be *well-defined*, or more informally, that the function "makes sense."

We now give several examples where we produce a rule on a domain where elements might have multiple representations, and then we check that it give a well-defined function. You'll get more practice with this on your homework.

Example 2.3. Let $\sim$ be the equivalence relation on $\mathbf{Z}$ given by $a \sim b$ iff $a - b$ is even. Then $\mathbf{Z} \to (\mathbf{Z}/\sim)$, $a \mapsto [a]$ is well-defined, because there is a *unique* way to write every integer in a way that can be input into our "rule" (i.e. we write a as a number, rather than any other sort of expression like a sum). Here, it doesn't matter that elements in $(\mathbf{Z}/\sim)$ can have multiple representations, because our rule "acts on" elements of $\mathbf{Z}$.

Example 2.4. Consider the "function" on $\mathbf{Q}$ given by $a/b \mapsto a^2/b^2$. We verify that this is actually well-defined, because our rule is given in terms of a specific representation of a rational number (and not the abstract rational number itself). To do so, we need to check that if $a/b = c/d$, then $a^2/b^2 = c^2/c^2$. If $c/d = a/b$ with $a, d \neq 0$, then by definition $ad = bc$, so $a^2d^2 = b^2c^2$, and we conclude that $a^2/b^2 = c^2/d^2$. So our function is well-defined: it does not depend on the specific representation of the fraction a/b.

Example 2.5. Let $\sim$ be the equivalence relation on $\mathbf{Z}$ given by $a \sim b$ iff $6|(a-b)$, and let $\sim'$ be the equivalence relation on $\mathbf{Z}$ given by $a \sim b$ iff $3|(a-b)$. Consider the "function" $(\mathbf{Z}/\sim) \to (\mathbf{Z}/\sim')$, putatively defined by $[a] \mapsto [a]'$. We will check that this is actually well-defined. Indeed, suppose $[b]$ is another representation of the equivalence class $[a]$. Then $[b] = [a]$, so $b \sim a$ and $6|(a-b)$. Now, once we apply our "rule" to $[a]$ and $[b]$, we get $[a]', [b]' \in (\mathbf{Z}/\sim')$. We claim that $[a]' = [b']$. Indeed, because $6|(a-b)$, then $3|(a-b)$, so $a \sim' b$ and $[a]' = [b]'$. Therefore, even though our "rule" $[a] \mapsto [a']$ is constructed depending on a specific representative of the equivalence class $[a]$, it does not matter which representative we pick, in the sense that different representatives of the same element of $(\mathbf{Z}/\sim)$ will produce the same output. Therefore our function is well-defined.

WARNING: It is easy at first to make the mistake of conflating a function's well-definedness with injectivity. The processes to check these properties look similar superficially, but there is a big difference. To check that a "rule" is well-defined is *part of the definition of a function* (e.g. a "ill-defined function" is not a function at all), but functions need not be injective. For instance, the well-defined function of Example 2.5 is not injective, because $[1] \neq [4]$ in $(\mathbf{Z}/\sim)$, but $[1]' = [4]'$ in $(\mathbf{Z}/\sim')$.

As a reminder, to check a "rule" f with inputs in A and outputs in B is a well-defined function, we need to show that if a and a' are any two equivalent ways of writing the *same element* in A, then the rule outputs the same element in B upon being applied to either a or a'. To check a function f is injective (so it is already well-defined, or this wouldn't even make sense to say), we check that if $a, a' \in A$ are such that $f(a) = f(a')$, then $a = a'$. Make sure you understand the difference!

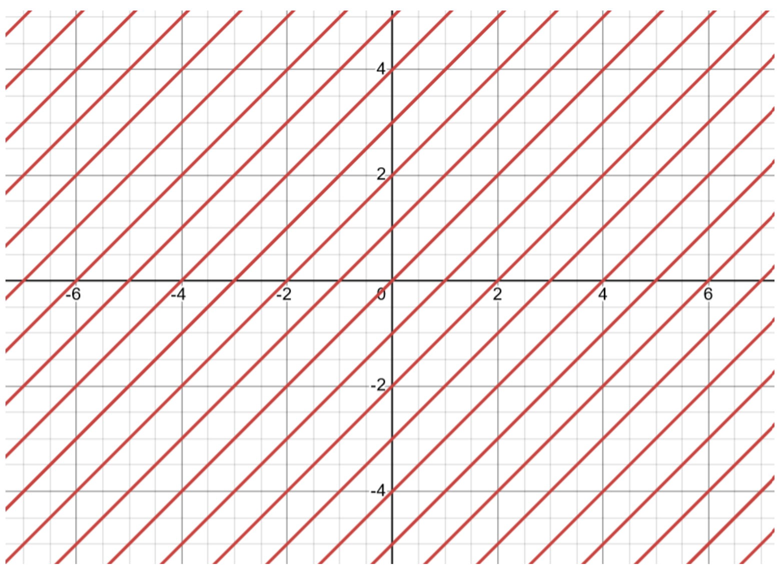
Example 2.6. In Example 1.7, we constructed a function $\mathbf{Z} \times (\mathbf{Z} - \{0\}) \to \mathbf{Q}$ via the rule $[(a,b)] \mapsto a/b$, which we claimed to be a bijection. On the other hand, because $\mathbf{Z} \times (\mathbf{Z} - \{0\})$ is a *quotient set*, where elements have multiple representations, we need to check that this function is well-defined. Indeed, if $[(a,b)]$ and $[(c,d)]$ are multiple ways of writing the same equivalence class, then by an argument similar to previous ones, we must have $(a,b) \sim (c,d)$, so $ad = bc$ by definition. Therefore the fractions a/b and c/d are equal in $\mathbf{Q}$ (two different ways of writing the same element), so our rule is well-defined: the outputs of $[(a,b)]$ and $[(c,d)]$ (which are different representatives of the same element) under our rule are equal in $\mathbf{Q}$.

Example 2.7. Here is an example from linear algebra. Consider the set $\mathbf{R}^2$, and put the equivalence relation $\sim$ on $\mathbf{R}^2$ where $(x,y) \sim (x',y')$ iff $x - x' = y - y'$ (if you have taken linear algebra before, you will recognize this as the quotient of $\mathbf{R}^2$ by the 1-dimensional subspace spanned by $(1,1)$). We claim that there is a natural bijection $f : (\mathbf{R}^2/\sim) \to \mathbf{R}$, given by $[(x,y)] \mapsto x-y$. We first must check that this rule is well-defined. Indeed, if $[(x,y)] = [(x',y')]$, then $(x,y) \sim (x',y')$, so by definition, $x-x' = y-y'$. Hence $x-y = x'-y'$, so the outputs of $[(x,y)]$ and $[(x',y')]$ under this rule are equal. Hence we have a well-defined function.

Surjectivity of f is easy: any $r \in \mathbf{R}$ is the image of $[(r,0)]$. For injectivity, note that if $[(x,y)]$ and $[(x',y')]$ have the same output under f, then $x-y = x'-y'$, so $x-x' = y-y'$. Hence $(x,y) \sim (x',y')$, so $[(x,y)] = [(x',y')]$, verifying injectivity of f.

WARNING: Once again, the verifications that f is well-defined and that f is injective seem very similar, but they are really quite different. The reason that injectivity of f seemed to use the exact same properties as the well-definedness of f is by coincidence: the equivalence relation $\sim$ was constructed to exactly make enough identifications on $\mathbf{R}^2$ to force the map to be both well-defined and injective (if you've taken abstract algebra before, you'll recognize this as "quotienting out by the kernel," where the kernel in question is that of the linear map $\mathbf{R}^2 \to \mathbf{R}, (x,y) \mapsto x-y$). In general, this need not happen, but you'll see the general idea on how to do this ("have $\sim$ make exactly the right identifications") on the homework.

Below is a graph of what this quotient map looks like geometrically.



Here, the equivalence classes of $\sim$ are precisely the diagonal lines, so these are the elements of $\mathbf{R}^2/\sim$. The map $f : (\mathbf{R}^2/\sim) \to \mathbf{R}$ is precisely the map sending a line to the x-coordinate of the point where it crosses the x-axis (think about this!). Put this way, it is much more apparent that f is a (well-defined) bijection. In fact, if g is the map $\mathbf{R}^2 \to \mathbf{R}$ given by $(x,y) \mapsto x-y$, then f is "induced from" g in the sense that the equivalence classes of $\sim$ (i.e. the elements of $(\mathbf{R}^2/\sim)$) are precisely the *fibers* of g. See Problem 6, HW6 for more about this construction.